

# The Dynamical Retardation Corrections to the Mass Spectrum of Heavy Quarkonia

**T.Kopaleishvili<sup>†</sup>, A.Rusetsky<sup>‡</sup>**

*High Energy Physics Institute, Tbilisi State University*

*9, University st., 380086, Tbilisi, Republic of Georgia*

<sup>†</sup> E-mail: root@hepitu.acnet.ge, root@presid.acnet.ge

<sup>‡</sup> E-mail: george@mdc.acnet.ge

## Abstract

A new version for the relativistic generalization of the wide class of static quark–antiquark confining potentials is suggested. The comparison of this approach with other ones, known in literature, is considered. With the use of Logunov–Tavkhelidze quasipotential approach the first–order retardation corrections to the heavy quarkonia mass spectrum are calculated. As expected, these corrections turn out to be small for all low–lying heavy meson states.

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# 1 Introduction

Within the framework of the constituent quark model it is quite natural to study the properties of bound  $q\bar{q}$  systems on the basis of Bethe–Salpeter equation. Despite the remarkable success in the quantitative description of the bound–state masses and formfactors, achieved with the use of instantaneous (static) interaction kernel in this equation [1–4], the completely relativistic approach to the problem is still lacking. From the physical point of view one can expect that the dynamical retardation corrections (i.e. the corrections, coming from the explicit dependence of the quark–quark interaction kernel on the relative energy variables) to the bound–state characteristics must be small for heavy quarkonia and may become significant in the light–quark sector. Moreover, one expects smooth static limit, when the masses of the constituent quarks tend to the infinity. In practice, however, the situation is more complicated, owing to the infrared–singular behaviour of the (phenomenological) confining quark–quark interaction kernels. Since at the present stage the exact derivation of such relativistic kernels directly from QCD is not known, the different prescriptions are assumed for the *ad hoc* relativistic generalization [5–11] of the static interquark confining potentials. As to the one–gluon exchange part of the interquark potential, which has proven to be significant in the quantitative description of meson data, it can be uniquely generalized to 4 dimensions with the use of the field–theoretical arguments. In difference with the one–gluon exchange potential, the ”relativization” of the confining potentials, in general, introduces a new mass parameter in the theory [7, 9, 10, 11], which may be fixed, using the additional constraints either directly on the relativistic counterpart of this potential [7, 9, 10, 11] or on the bound–state equation [8]. Neither of these constraints can be preferred from the physical point of view, rendering the identification of the dynamical retardation effect ambiguous.

The existence of the additional free parameter in the theory stems from the necessity of the infrared regularization of the ”confining” kernel in 4 dimensions. As a result, the smooth static limit, in general, does not exist [9, 11] and the dynamical retardation corrections to the bound–state characteristics turn out to

be large in the heavy-quark sector [9, 11], rendering doubtful even the concept of the confining interaction in 4 dimensions [9]. Moreover, the straightforward relativistic generalizations for some widely used confining potentials (e.g. for the oscillator potential) seem do not possess the discrete spectrum [12] in the entire energy region [11]. Thus, the further investigation of the problem in the framework of Bethe–Salpeter equation with the relativistic kernels is needed in order to clarify the notion of the confining force in 4 dimensions. In other words, one has to answer the question, whether there exist relativistic kernels which are compatible with confinement, lead to a smooth static limit and, therefore, can be expected to be obtained from QCD with the use of nonperturbative methods.

## 2 The Relativistic Generalization of the Static Confining Potentials

In the present work we suggest a new prescription for the relativistic generalization of the wide class of static confining potentials, where some difficulties which are inherent for the prescriptions, suggested earlier [5-10], are presumably avoided. In order to clarify the physical meaning of this prescription, let us consider the case of two space–time dimensions. We start from the power–law potentials  $V_\alpha(r) = r^\alpha$ . The corresponding Bethe–Salpeter kernel in two dimensions has the form:

$$K_\alpha(x) = K_\alpha(x_0, x_1) = \delta(x_0)V_\alpha(|x_1|) = \delta(nx)(-x^2)^{\alpha/2} \quad (1)$$

where  $x^2 = x_0^2 - x_1^2$  and  $n = (1, 0)$ . We assume that the relativistic generalization of this expression is the "Lorentz–average" over all possible directions of the unit vector  $n$  which can be obtained from the vector  $(1, 0)$  by the proper Lorentz transformations. In 2 space–time dimensions this vector transforms as  $n'_0(\varphi) = n_0 \cosh \varphi - n_1 \sinh \varphi$ ,  $n'_1(\varphi) = n_1 \cosh \varphi - n_0 \sinh \varphi$ ,  $\varphi = \frac{v}{c}$  and  $-\infty < \varphi < \infty$  and the group–invariant measure is  $d\varphi$ . Then the "average" over all possible directions of  $n$  can be defined as follows:

$$\bar{K}_\alpha(x) = (-x^2)^{\alpha/2} \langle \delta(nx) \rangle_n = (-x^2)^{\alpha/2} \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} d\varphi \delta(x_0 n'_0(\varphi) - x_1 n'_1(\varphi)) \quad (2)$$

where  $\mathcal{N}$  is an arbitrary Lorentz-invariant normalization constant. The integral in the r.h.s. of eq. (2) can be easily evaluated:

$$\int_{-\infty}^{\infty} d\varphi \delta(x_0 n'_0(\varphi) - x_1 n'_1(\varphi)) = \theta(-x^2)(-x^2)^{-1/2} \quad (3)$$

The constant  $\mathcal{N}$  is fixed from the condition that in the static limit the kernel (2) reduces to its static counterpart  $V_\alpha(r)$ :  $\int_{-\infty}^{\infty} dx_0 \bar{K}_\alpha(x_0, x_1) = V_\alpha(|x_1|)$ . This gives:

$$\bar{K}_\alpha(x) = \frac{\Gamma(1 + \frac{\alpha}{2})}{\sqrt{\pi} \Gamma(\frac{1}{2} + \frac{\alpha}{2})} \theta(-x^2)(-x^2)^{(\alpha-1)/2} \quad (4)$$

Thus, the relativistic kernel (4) in two dimensions can be interpreted merely as the static potential, averaged over all possible directions of the quantization axe, which are equally acceptable from the physical point of view. For the case of 4 space-time dimensions the straightforward evaluation of the integral over the group-invariant measure in analogy with the eqs. (2)–(3) is not possible. Despite this the "average" of the static kernel (1) over all possible directions of the unit vector  $n$  can be defined for this case as well, and we again arrive at the expression (4) (for the details see Appendix A). Note that the Coulombic interaction in 4 dimensions ( $\alpha = -1$ ) is excluded in the expression (4).

Having obtained the expression (4), we can forget about its "derivation" and consider (4) merely as an *anzats* for the relativistic generalization of the power-law potentials  $V_\alpha(r) = r^\alpha$ . Further, we consider the exponential static potential  $V_{exp}(r) = e^{-\mu r}$ . Expanding this potential in powers of  $r$  and using eq. (4) for each term of this sum, we obtain the prescription:

$$e^{-\mu r} \rightarrow \theta(-x^2) \left( -\frac{\mu}{2} J_0(\mu \sqrt{-x^2}) + \frac{1}{\pi} (-x^2)^{-1/2} {}_1F_2\left(1; \frac{1}{2}, \frac{1}{2}; -\frac{\mu^2 x^2}{4}\right) \right) \quad (5)$$

where  $J_\nu$  and  ${}_pF_q$  denote, respectively, the Bessel and hypergeometric functions. Eq. (5) enables one to apply the procedure of the relativistic generalization to the wide class of potentials which can be written in the following form:

$$V(r) = \int d\mu C(\mu) e^{-\mu r} \quad (6)$$

where  $C(\mu)$  must obey to the certain conditions in order to provide the convergence of the integral over  $d\mu$  in the relativistic case.

Now we discuss the particular case of linear confinement ( $\alpha = 1$ ) in detail. Note that the ansatz similar to (4) for the gluon propagator was employed earlier in ref. [13, 23]. The Fourier transform of eq. (4) for  $\alpha = 1$  reads:

$$\bar{K}_1(q) = -4\pi \left( \frac{1}{(q_0^2 - (|\vec{q}| + i0)^2)^2} + \frac{1}{(q_0^2 - (|\vec{q}| - i0)^2)^2} \right) \quad (7)$$

i.e., in other words, in order to obtain the kernel (7) in the ladder approximation, the principal-value prescription must be used in the gluon propagator instead of the familiar causal one. One can find here a close analogy with the 2-dimensional QCD (see, e.g. [14-16]), where the gluon exchange is definitely known to confine quarks as well as with the gluon propagator from ref. [21]. As it is well known [23], due to the principal-value prescription in (7) the most severe infrared singularities in the equations for the Green's functions are avoided. To demonstrate this, we consider the Schwinger-Dyson equation for the quark propagator written in the Feynman gauge:

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + ig^2 C_F \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu S(p-q) \Gamma^\mu(p-q, p) D(q) = \\ &= S_0^{-1}(p) + ig^2 C_F \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu (S(p-q) \Gamma^\mu(p-q, p) - S(p) \Gamma^\mu(p, p)) D(q) + \\ &\quad + ig^2 C_F \gamma_\mu S(p) \Gamma^\mu(p, p) \int \frac{d^4 q}{(2\pi)^4} D(q) \end{aligned} \quad (8)$$

where  $S(p)$  and  $S_0(p)$  are full and free fermion propagators,  $\Gamma^\mu(k, p)$  is the dressed quark-gluon vertex function,  $D(q)$  is the dressed gluon propagator,  $g$  is the quark-gluon coupling constant and  $C_F$  is the quadratic Casimir operator in the fundamental representation. If the principal-value ansatz (7) is used for  $D(q)$ , then the last term in eq. (8) is finite and we obtain the subtracted form of the Schwinger-Dyson equation, where the infrared singularity is softened. In difference with this, in the Euclidean formulation of Schwinger-Dyson equations, which implies the causal prescription for all propagators after the continuation to the Minkowski space, the last term in eq. (8) diverges and requires the infrared regularization [17]. Consequently, the relativistic generalization of the static interquark interaction kernels introduces an additional mass parameter when the causal prescription is used in these kernels or, equivalently, when one works in the Euclidean space

from the beginning. However, no such parameter is needed, when one uses the principal-value prescription (7) or, in general, the prescriptions (4)–(6). In the absence of the additional free mass parameter one expects the existence of the smooth static limit for the calculated meson observables, while the corrections to the static limit should allow for the unambiguous evaluation.

### 3 First–Order Quasipotential for the $q\bar{q}$ Systems

The formulated approach for the construction of the relativistic confining kernels below we apply to the calculation of the dynamical retardation corrections to the bound  $q\bar{q}$  system masses. The static confining potential was taken to be linear + constant term:  $V_c(r) = kr + c$  and the spin structure was chosen to be the equal-weight mixture of scalar and the fourth component of vector:  $\hat{O}_c = \frac{1}{2}(I_1 \otimes I_2 + \gamma_1^0 \otimes \gamma_2^0)$  which is perhaps the simplest choice from the more general ones [1-4] and provides the existence of the stable discrete energy levels. The one-gluon exchange part of the potential was neglected since we were interested in the retardation corrections, coming from the confining part of the potential. Further, for the simplicity, we have restricted ourselves to the equal-mass case. We have used the Logunov–Tavkhelidze quasipotential approach [18] in order to reduce the original, 4-dimensional Bethe–Salpeter equation to the 3-dimensional one, which can be numerically solved with the use of the conventional mathematical methods.

The quasipotential equation for the 3-dimensional equal-time wave function  $\tilde{\varphi}(\vec{p})$  (in c.m.f.) is written in the following form [10, 11]:

$$[M_B - h_1(\vec{p}) - h_2(-\vec{p})]\tilde{\varphi}(\vec{p}) = -i\gamma_1^0\gamma_2^0 \frac{4}{3} \int \frac{d^3\vec{q}}{(2\pi)^3} \tilde{V}(M_B; \vec{p}, \vec{q}) \tilde{\varphi}(\vec{q}) \quad (9)$$

where  $M_B$  is the mass of the bound  $q\bar{q}$  system and  $h_i = \vec{\alpha}_i \vec{p}_i + m\gamma_i^0$ ,  $m$  being the mass of the constituent quark. Keeping in mind that the retardation corrections for heavy quarkonia are expected to be small, further we restrict ourselves to the first-order quasipotential formalism, where  $\tilde{V}$  is given by the following expression [10, 11]:

$$\tilde{V}^{(1)}(M_B; \vec{p}, \vec{q}) = \langle \vec{p} | \underline{\tilde{G}}_0^{-1} G_0 \widetilde{K} G_0 \underline{\tilde{G}}_0^{-1} | \vec{q} \rangle \quad (10)$$

Here  $G_0$  is the free two-fermion Green's function,  $K$  is the Bethe-Salpeter equation kernel and the procedure  $\tilde{A}$  for any operator  $A$  is defined as follows:

$$\tilde{A}(P; \vec{p}, \vec{q}) = \int \frac{dp_0}{2\pi} A(P; p, q) \frac{dq_0}{2\pi} \quad (11)$$

and

$$\begin{aligned} \tilde{G}_0 &= \tilde{G}_0 \gamma_1^0 \gamma_2^0 \Pi; \quad \Pi = (\Lambda_1^{(+)} \Lambda_2^{(+)} - \Lambda_1^{(-)} \Lambda_2^{(-)}) \gamma_1^0 \gamma_2^0; \\ \Lambda_i^{(\pm)} &= \frac{w_i \pm h_i}{2w_i}; \quad w_i = (m^2 + \vec{p}_i^2)^{\frac{1}{2}} \end{aligned} \quad (12)$$

The partial-wave decomposition of the eq. (9) is done with the use of the standard technique [2, 10, 11]. In the wave function  $\tilde{\varphi}(\vec{p})$  we retain only the contribution from the "double-positive" component  $\tilde{\varphi}^{(++)}(\vec{p}) = \Lambda_1^{(+)} \Lambda_2^{(+)} \tilde{\varphi}(\vec{p})$ , as the role of the negative-energy component is shown to be small provided the stable solutions exist [2]. Using the substitution

$$\tilde{\varphi}^{(++)}(\vec{p}) = \left( \frac{w+m}{2w} \right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}_1 \vec{p}}{w+m} \end{pmatrix} \otimes \left( \frac{w+m}{2w} \right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{-\vec{\sigma}_2 \vec{p}}{w+m} \end{pmatrix} \tilde{\chi}^{(+)}(\vec{p}) \quad (13)$$

and decomposing the Pauli spinor  $\tilde{\chi}^{(+)}(\vec{p})$  into partial waves

$$\tilde{\chi}^{(+)}(\vec{p}) = \sum_{LSJM_J} \langle \vec{n} | LSJM_J \rangle \tilde{R}_{LSJ}^{(+)}(p); \quad \vec{n} = \frac{\vec{p}}{p}; \quad S = 0, 1 \quad (14)$$

from the eq. (9) we obtain

$$\begin{aligned} [M_B - 2w(p)] \tilde{R}_{LSJ}^{(+)}(p) &= \frac{4}{3} \int_0^\infty q^2 dq \left( \tilde{V}_L^{(+)}(p, q) \frac{1}{2} \left( 1 + \frac{m^2}{w(p)w(q)} \right) + \right. \\ &+ (L - J) \frac{4J(J+1)}{(2J+1)^2} (\tilde{V}_{J-1}^{(+)}(p, q) - \tilde{V}_{J+1}^{(+)}(p, q)) \frac{(w(p)-m)(w(q)-m)}{4w(p)w(q)} \left. \right) \tilde{R}_{LSJ}^{(+)}(q) \end{aligned} \quad (15)$$

where, for the case of local quasipotential

$$\tilde{V}_L^{(+)}(p, q) = \int r^2 dr j_L(pr) \tilde{V}_c(r; M_B) j_L(qr) \quad (16)$$

$j_L$  being the spherical Bessel function and we, as in ref. [10, 11], have neglected small mixing between  $L = J - 1$  and  $L = J + 1$  states.

After doing some algebra with the projection operators the positive-energy component of the first-order quasipotential, corresponding to the interaction kernel (4) with the use of eqs. (10), (11) can be written in the following form:

$$\tilde{V}^{(+)}(M_B; \vec{p}, \vec{q}) = \int d^3 \vec{x} e^{-i(\vec{p}-\vec{q})\vec{x}} \frac{\Gamma \left( 1 + \frac{\alpha}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{2} + \frac{\alpha}{2} \right)} r^\alpha \times$$

$$\times \int_{-1}^1 d\tau (1 - \tau^2)^{(\alpha-1)/2} (\theta(\tau) e^{i(M_B - w(p) - w(q))r\tau} + \theta(-\tau) e^{-i(M_B - w(p) - w(q))r\tau}) \quad (17)$$

Neglecting relativistic corrections in the exponentials  $M_B - w(p) - w(q) = M_B - 2m + O(\frac{1}{m}) = -\epsilon_B$ , as well as the imaginary part of this expression in analogy with the refs. [10, 11, 19], we arrive at the local first-order quasipotential:

$$\begin{aligned} \tilde{V}^{(+)}(r; \epsilon_B) &= \frac{\Gamma\left(1 + \frac{\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2} + \frac{\alpha}{2}\right)} r^\alpha \int_{-1}^1 d\tau (1 - \tau^2)^{(\alpha-1)/2} \cos \epsilon_B r \tau = \\ &= \left|\frac{2r}{\epsilon_B}\right|^{\alpha/2} \Gamma\left(1 + \frac{\alpha}{2}\right) J_{\alpha/2}(|\epsilon_B r|) \end{aligned} \quad (18)$$

Hence, the relativistic generalization of the static potential  $V_c(r) = kr + c$  gives the first-order local quasipotential

$$\tilde{V}_c(r, \epsilon_B) = k \frac{\sin \epsilon_B r}{\epsilon_B} + c J_0(\epsilon_B r) \quad (19)$$

which accounts for the retardation effect and reduces to  $V_c(r)$  in the limit  $\epsilon_B \rightarrow 0$ .

As it can be seen from eq. (19), the account for the dynamical retardation effect in the case of pure linear potential effectively leads to the colour screening at an intermediate distances. It should be pointed out that such behaviour qualitatively agrees with the results of calculations for the unquenched lattice fermions in QCD [20]. At a larger distances the deviation of the retarded potential from the static one becomes significant, and one can no further rely on the first-order calculations. Note, however, that for the case of heavy quarkonia the latter difficulty causes no trouble since the wave function of the  $q\bar{q}$  bound system in this case rapidly vanishes with the increase of  $r$  and, therefore, does not "feel" the oscillating "tail" of the potential at a large distances.

## 4 Results and Discussion

It is not obvious from the beginning whether the potential (19) leads to the discrete energy levels due to its oscillating behaviour at  $r \rightarrow \infty$ . Let us, therefore consider the equation (15) with the potential (19) in detail. Passing to the nonrelativistic limit and neglecting for a moment the "constant" term in (19),

proportional to  $c$ , in the configuration space we obtain the following differential equation:

$$f''(z) + (a \cos 2z + b)f(z) = 0 \quad (20)$$

where  $f(r) = rR(r)$ ,  $R(r) \equiv R_0(r)$  being the radial wave function of the bound state in the configuration space (for simplicity we assume the angular momentum,  $L = 0$ ),  $z = \frac{1}{2}((M_B - 2m)r - \pi/2)$ ,  $a = \frac{k(M_B - 2m)}{3m^3}$ ,  $b = \frac{(M_B - 2m)}{4m^3}$ , and the boundary conditions imposed on  $f(z)$  are  $f(-\frac{\pi}{4}) = 0$  and  $f(+\infty) = 0$ .

The equation (20) has been extensively studied in the mathematical physics (see, e.g. [22]). We shall remind some results of this investigation. Namely, if  $f_1(z)$  is the particular solution of the eq. (20) with the following initial conditions:

$$f_1(0) = 1, \quad f_1'(0) = 0 \quad (21)$$

and

$$\cosh 2\pi\mu = f_1(\pi)$$

Then the general solution of the equation (20) has the form:

$$f(z) = \begin{cases} C_1 e^{2\mu z} \varphi_1(z) + C_2 e^{-2\mu z} \varphi_2(z); & \cosh 2\pi\mu > 1 \\ (C_1 \cos 2\nu z + C_2 \sin 2\nu z) \varphi_1(z) + (C_2 \cos 2\nu z - C_1 \sin 2\nu z) \varphi_2(z); & |\cosh 2\pi\mu| < 1; \quad \mu = i\nu \\ C_1 e^{2\rho z} \varphi_1(z) + C_2 e^{-2\rho z} \varphi_2(z); & \cosh 2\pi\mu < -1; \quad \mu = \rho + \frac{i}{2} \end{cases} \quad (22)$$

$\varphi_1(z)$  and  $\varphi_2(z)$  being the periodic functions in  $z$  with the period  $\pi$ .

Due to the fact that we consider the equation (20) on the semi-infinite interval  $-\frac{\pi}{4} \leq z < +\infty$ , it is possible to find the normalizable solutions decreasing exponentially at  $z \rightarrow +\infty$  ( $C_1 = 0$  and  $|\cosh 2\pi\mu| > 1$ , eq. (22)). The eigenvalue condition then reads as:

$$\varphi_2\left(-\frac{\pi}{4}\right) = 0, \quad |\cosh 2\pi\mu| > 1 \quad (23)$$

Thus, the equation (20), despite the oscillating behaviour of the potential at the spatial infinity, allows for the discrete spectrum provided  $|\cosh 2\pi\mu| > 1$ ,

corresponding to the condition  $M_B - 2m < 0$  in the limit  $|M_B - 2m| \ll 2m$ . Adding the constant term, proportional to  $c$ , it is natural to suppose that, for a small  $|M_B - 2m|$  the discrete energy levels exist for  $M_B - 2m - \frac{4}{3}c < 0$ . Thus the potential (19) in the nonrelativistic limit acts like the potential well. Note that the similar potential (the rising potential screened at large distances,  $r > 1$  Fm) was successfully used for the description of meson spectrum in the framework of coupled Dyson–Schwinger and Bethe–Salpeter equations e.g. in ref. [24]. Therefore we expect that the equation (15) with the potential (19) gives the reasonable description of the low-lying meson states.

At the next step we have attempted to solve the eq. (15) numerically, expanding the unknown radial wave function  $\tilde{R}_{LSJ}^{(+)}(p)$  in the complete orthonormalized basis of the nonrelativistic oscillator wave functions [1, 2, 10, 11]

$$\tilde{R}_{LSJ}^{(+)}(p) = p_0^{-3/2} \sum_{n=0}^{\infty} c_{nLSJ}^{(+)} R_{nL}(p/p_0) \quad (24)$$

where

$$R_{nL}(z) = \left( \frac{2\Gamma\left(n + L + \frac{3}{2}\right)}{\Gamma(n+1)} \right)^{\frac{1}{2}} \frac{1}{\Gamma\left(L + \frac{3}{2}\right)} z^L \exp\left(-\frac{1}{2}z^2\right) {}_1F_1\left(-n, L + \frac{3}{2}, z^2\right) \quad (25)$$

and  $p_0$  is an arbitrary scale parameter. Inserting (24) in the equation (15) and truncating the sum at some fixed value  $N_{max}$ , we arrive at a system of linear algebraic equations for the coefficients  $c_{nLSJ}^{(+)}$ . If the procedure converges with the increase of  $N_{max}$ , the eigenvalues  $M_B$  are determined from this system of equations. The calculations show that the final results do not depend on the scale parameter  $p_0$ , but the appropriate choice of this parameter leads to the faster convergence of the series (24). It should be stressed that if the solution of the equation (15) does not exist, this reveals in the divergence of the procedure with the increase of  $N_{max}$  despite the fact that the potential matrix elements are calculated in the exponentially damping wave function basis.

Since the potential (19) depends on the unknown binding energy,  $\epsilon_B = 2m - M_B$ , of the  $q\bar{q}$  system, the equation (15) is solved with the use of the iteration method. Namely, we solve the equation with the static potential,  $V_c(r) = kr + c$  and determine the eigenvalues  $M_B^{(st)}$ . At the next step these static values are

substituted into the potential (19) in order to determine the corrected spectrum which, in its turn, is used, as an input, in the next iteration. We have checked that, typically after 10–15 steps, the iteration procedure converges for the most low-lying heavy quarkonia energy levels.

In the table 1 the results of calculations of the dynamical retardation corrections to the heavy quarkonia mass spectrum are presented. In these calculations the parameters  $k$  and  $c$  were taken to be  $k = 0.21 \text{ GeV}^2$ ,  $c = -1.0 \text{ GeV}$ . The constituent quark masses were chosen to be  $m_c = 1.72 \text{ GeV}$  and  $m_b = 5.10 \text{ GeV}$  in order to fit  $J/\psi$  and  $\Upsilon$  masses. As we see from table 1, this set of parameters gives the reasonable description of heavy meson mass spectrum in the static approximation. As expected, the dynamical retardation corrections turn out to be small (typically a few percent) for all low-lying quarkonia states given in this table.

As we see, the present approach to the relativistic generalization of the static confining potential does not suffer from some of difficulties inherent of the conventional ones [5–10]. Namely, an additional mass scale parameter does not appear in the "relativized" counterpart of the static potential. Moreover, this approach possesses a smooth static limit and yields small corrections due to the retardation in the heavy quarkonia mass spectrum. This, unlike the results of refs. [9, 11], does not manifestly contradict with our expectation at least in the heavy-quark sector. An important remark, however, should be made, concerning the approach suggested in the present paper. Namely, the kernels of the type considered here (eqs. (4), (5)) can never be obtained in the Euclidean formulation of QCD (see eq. (7)). Consequently, such fundamental properties, as microcausality and unitarity, do not necessarily continue to hold in this approach. Of course, one can appeal again to the 2-dimensional QCD, where, in spite of the principal-value prescription used in the gluon propagator, the uncoloured current-current correlation functions are known to have the correct analytical structure [16]. Note, however, that in 2 dimensions the gluon field is not a dynamical one [15], unlike the case of 4 dimensions, where the different components of the gluon field (dynamical and nondynamical) may be subject to the different boundary conditions. Here one

finds an analogy with the result obtained in the ref. [21] where, in particular, it was demonstrated that the ghost degrees of freedom (in the covariant gauge) can be eliminated from the theory at the cost of Feunman's boundary conditions in the propagator for timelike and longitudinal gluons, being replaced by the principal-value prescription. An alternative point of view was presented in ref. [23], where the quantization procedure for the gauge field is carried out with the account of the fact that this field is never observed in *in*- or *out*- states. As a result, the principal-value prescription emerges in the propagator for all components of the gauge field instead of the conventional Feunman one. Consequently, the anzats (4), corresponding to the "minimal" relativistic generalization of the static interquark interactions, can be regarded as an appropriate candidate for the truly confining interquark interaction kernel in 4 dimensions.

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## Appendix A

Below we define the "average" of the static kernel (1) over all possible directions of the unit vector  $n$  in the 4-dimensional Minkowski space. This approach can be applied to the case of any space-time dimensions and, in particular, to the case of 2 dimensions, considered in the text.

We start from the Lorentz-invariant regularization of the "average" of  $\delta$ -function, entering the expression (1):

$$< \delta(nx) >_n = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} < e^{i\lambda(nx) - \epsilon\lambda^2} (\theta(x^2) + \theta(-x^2)) >_n \equiv \bar{\delta}_+ + \bar{\delta}_- \quad (\text{A.1})$$

Let us consider first the case  $x^2 > 0$

$$\bar{\delta}_+ = \theta(x^2)(x^2)^{-1/2} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} < e^{i\lambda(n\eta_+) - \epsilon\lambda^2} >_n; \quad \eta_+^\mu = x^\mu / \sqrt{x^2}; \quad \eta_+^2 = 1 \quad (\text{A.2})$$

Expanding  $e^{i\lambda(n\eta_+)}$  in powers of  $\lambda$  and defining the "average" in a familiar manner:

$$< n_{\mu_1} \cdots n_{\mu_k} >_n = 0 \quad \text{for } k \text{ odd}$$

$$\langle 1 \rangle_n = 1; \quad \langle n_{\mu_1} n_{\mu_2} \rangle_n = \frac{1}{4} g_{\mu_1 \mu_2}; \quad etc \quad (A.3)$$

we obtain

$$\langle e^{i\lambda(n\eta_+)} \rangle_n = 1 + \frac{(i\lambda)^2}{2!} \frac{1}{4} + \dots = F_+(\lambda) \quad (A.4)$$

In order to calculate  $F_+(\lambda)$  we consider the 4-dimensional Euclidean space and define the average:

$$\langle e^{i\lambda(n^E \eta_+^E)} \rangle_{n^E} = \frac{1}{\pi^2} \int d^4 n^E \delta(n^{E2} - 1) e^{i\lambda(n^E \eta_+^E)}; \quad n^{E2} = \eta_+^{E2} = 1 \quad (A.5)$$

Using the relations

$$\begin{aligned} \langle n_{\mu_1}^E \dots n_{\mu_k}^E \rangle_{n^E} &= 0 \quad \text{for } k \text{ odd} \\ \langle 1 \rangle_{n^E} &= 1; \quad \langle n_{\mu_1}^E n_{\mu_2}^E \rangle_{n^E} = \frac{1}{4} \delta_{\mu_1 \mu_2}; \quad etc \end{aligned} \quad (A.6)$$

it is easy to ensure that (A.5) coincides with  $F_+(\lambda)$ . Passing in the integral in (A.5) to the angular variables and substituting the result into (A.2), we finally obtain

$$\bar{\delta}_+ = \theta(x^2)(x^2)^{-1/2} \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\lambda \int_{-1}^1 dy \sqrt{1-y^2} e^{i\lambda y - \epsilon \lambda^2} = \theta(x^2)(x^2)^{-1/2} \frac{1}{2\pi} \quad (A.7)$$

The case  $x^2 < 0$  can be considered in a similar way. However, due to the change of sign  $\eta_-^2 = -1$ , where  $\eta_-^\mu = x^\mu / \sqrt{-x^2}$ , in the integrand  $e^{i\lambda y}$  is replaced by  $e^{\lambda y}$

$$\bar{\delta}_- = \theta(-x^2)(-x^2)^{-1/2} \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\lambda \int_{-1}^1 dy \sqrt{1-y^2} e^{\lambda y - \epsilon \lambda^2} \quad (A.8)$$

and, as a result,  $\bar{\delta}_-$  diverges in the limit  $\epsilon \rightarrow 0$ . Including this divergent (Lorentz-invariant) factor in the overall normalization constant  $\mathcal{N}$ , we see, that only the term  $\bar{\delta}_-$  survives in the limit  $\epsilon \rightarrow 0$ . Finally, substituting (A.8) into the eq. (2) and determining the normalization constant in the static limit, we again arrive at the expression (4) for the relativistic Bethe-Salpeter kernel.

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Mesons	$J^{PC}$	$N^{2S+1}L_J$	1	2	3	4
$c\bar{c}$						
$\eta_c(2.980)$	$0^{-+}$	$1^1S_0$	3.095	3.204	0.109	$3.5 \cdot 10^{-2}$
$\eta'_c(3.590)$	$0^{-+}$	$2^1S_0$	3.690	3.739	0.049	$1.3 \cdot 10^{-2}$
$J/\psi(3.097)$	$1^{--}$	$1^3S_1$	3.096	3.204	0.108	$3.5 \cdot 10^{-2}$
$\psi'(3.686)$	$1^{--}$	$2^3S_1$	3.691	3.742	0.051	$1.3 \cdot 10^{-2}$
$h_{c1}(3.526)$	$1^{+-}$	$1^1P_1$	3.445	3.460	0.014	$4.2 \cdot 10^{-3}$
$\chi_{c0}(3.414)$	$0^{++}$	$1^3P_0$	3.445	3.460	0.014	$4.2 \cdot 10^{-3}$
$\chi_{c1}(3.511)$	$1^{++}$	$1^3P_1$	3.445	3.460	0.014	$4.2 \cdot 10^{-3}$
$\chi_{c2}(3.556)$	$2^{++}$	$1^3P_2$	3.445	3.461	0.014	$4.2 \cdot 10^{-3}$
$b\bar{b}$						
$\eta_b$	$0^{-+}$	$1^1S_0$	9.463	9.619	0.156	$1.7 \cdot 10^{-2}$
$\eta'_b$	$0^{-+}$	$2^1S_0$	9.899	9.966	0.067	$6.8 \cdot 10^{-3}$
$\Upsilon(9.460)$	$1^{--}$	$1^3S_1$	9.463	9.619	0.156	$1.7 \cdot 10^{-2}$
$\Upsilon'(10.023)$	$1^{--}$	$2^3S_1$	9.899	9.966	0.067	$6.8 \cdot 10^{-3}$
$\Upsilon''$	$1^{--}$	$1^3D_1$	9.938	9.993	0.055	$5.5 \cdot 10^{-3}$
$\Upsilon'''(10.355)$	$1^{--}$	$3^3S_1$	10.250	10.255	0.005	$5.1 \cdot 10^{-4}$
$\Upsilon^{IV}$	$1^{--}$	$2^3D_1$	10.277	10.291	0.014	$1.4 \cdot 10^{-3}$
$h_{b1}$	$1^{+-}$	$1^1P_1$	9.720	9.835	0.115	$1.2 \cdot 10^{-2}$
$\chi_{b0}(9.860)$	$0^{++}$	$1^3P_0$	9.720	9.835	0.115	$1.2 \cdot 10^{-2}$
$\chi_{b1}(9.892)$	$1^{++}$	$1^3P_1$	9.720	9.835	0.115	$1.2 \cdot 10^{-2}$
$\chi_{b2}(9.913)$	$2^{++}$	$1^3P_2$	9.720	9.835	0.115	$1.2 \cdot 10^{-2}$
$h'_{b1}$	$1^{+-}$	$2^1P_1$	10.097	10.109	0.013	$1.3 \cdot 10^{-3}$
$\chi'_{b0}(10.232)$	$0^{++}$	$2^3P_0$	10.097	10.109	0.013	$1.3 \cdot 10^{-3}$
$\chi'_{b1}(10.255)$	$1^{++}$	$2^3P_1$	10.097	10.109	0.013	$1.3 \cdot 10^{-3}$
$\chi'_{b2}(10.268)$	$2^{++}$	$2^3P_2$	10.097	10.109	0.013	$1.3 \cdot 10^{-3}$

Table 1. The dynamical retardation corrections to the heavy quarkonia mass spectrum. 1) The meson mass in the static approximation,  $M_B^{(st)}$  (GeV), 2) The meson mass with an account of the retardation effect,  $M_B^{(ret)}$  (GeV), 3) The size of the retardation correction,  $M_B^{(ret)} - M_B^{(st)}$  (GeV), 4)  $|M_B^{(ret)} - M_B^{(st)}|/M_B^{(st)}$ .